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# Upper minus total domination in small-degree regular graphs[☆]

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**Abstract**

A function $f: V(G) \to \{-1, 0, 1\}$ defined on the vertices of a graph $G$ is a minus total dominating function (MTDF) if the sum of its function values over any open neighborhood is at least one. An MTDF $f$ is minimal if there does not exist an MTDF $g: V(G) \to \{-1, 0, 1\}$, $f \neq g$, for which $g(v) \leqslant f(v)$ for every $v \in V(G)$. The weight of an MTDF is the sum of its function values over all vertices. The minus total domination number of $G$ is the minimum weight of an MTDF on $G$, while the upper minus domination number of $G$ is the maximum weight of a minimal MTDF on $G$. In this paper we present upper bounds on the upper minus total domination number of a cubic graph and a 4-regular graph and characterize the regular graphs attaining these upper bounds.


*MSC:* 05C69

*Keywords:* Minus total domination; Regular graph; Bounds

## 1. Introduction

All graphs considered here are finite, undirected, and simple. For standard graph theory terminology not given here, we refer to [6]. Specifically, let $G = (V, E)$ be a graph with *vertex set* $V$ and *edge set* $E$. The *order* of $G$ is given by $n = |V(G)|$. For a vertex $v$ in $V$, the *open neighborhood* of $v$ is $N(v) = \{u \in V \mid uv \in E\}$ and the *closed neighborhood* of $v$ is $N[v] = \{v\} \cup N(v)$. For a subset $S \subseteq V$, the open neighborhood of $S$ is $N(S) = \bigcup_{v\in S} N(v)$ and the closed neighborhood of $S$ is $N[S] = \bigcup_{v\in S} N[v]$. $G[S]$ denotes the subgraph of $G$ induced by $S$. The degree of $v$ in $G$ is denoted by $d(v)$. If $d(v)$ is odd, then $v$ is called an *odd vertex* of $G$. A graph $G$ is called *k-regular* if $d(v) = k$ for all $v \in V$. In particular, 3-regular graphs are also referred as *cubic graphs*. For a subset $S \subseteq V$, we use $d_S(v)$ denote the number of vertices in $S$ that are adjacent to $v$. For disjoint subsets $U$ and $W$ of vertices, we let $e(U, W)$ denote the number of edges between $U$ and $W$.

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A set $S \subseteq V(G)$ is a *total dominating set* of a graph $G$ if every vertex in $V$ is adjacent to a vertex in $S$, that is, $N(S) = V$. Every graph without isolated vertices has a total dominating set, since $S = V$ is such a set. The *total domination number* of $G$, denoted by $\gamma_t(G)$, is the minimum cardinality of a total dominating set. Total domination in graphs is introduced by Cockayne et al. and is now well studied in graph theory (see, for example, [6]).

For a real-valued function $f: V \to R$, the weight of $f$ is $w(f) = \sum_{v\in V} f(v)$, and for $S \subseteq V$, we define $f(S) = \sum_{v\in S} f(v)$, so $w(f) = f(V)$. For a vertex $v$ in $V$, we denote $f(N(v))$ by $f[v]$ for notational convenience. Let $f: V \to \{0, 1\}$ be a function which assigns to each vertex of a graph without isolated vertices an element in the set $\{0, 1\}$. Then, $f$ is called *total dominating function* (TDF) if for every $v \in V$, $f[v] \geqslant 1$.

Let $f: V \to \{-1, 0, 1\}$ be a function which assigns to each vertex of $G$ an element of the set $\{-1, 0, 1\}$. The function $f$ is defined in [3] to be *minus dominating function* (MDF) of $G$ if $\sum_{u\in N[v]} f(u) \geqslant 1$ for every $v \in V$. The *minus domination number*, denoted by $\gamma^-(G)$, of $G$ is the minimum weight of an MDF on $G$. Minus domination has been studied in [1–3,8–17] and elsewhere. If we only allow the weights $-1$ and 1, then this is a well-known *signed domination* which is first introduced in [4]. Zelinka [18] develops an analogous theory for signed total domination that arises when we simply change "closed" neighborhood in the definition of signed domination to "open" neighborhood. The parameter is studied by Henning in [7].

Recently, Harris and Hattingh [5] introduce the concept of minus total domination, and show that the decision problem for the minus total domination number of a graph is NP-complete, even when restricted to bipartite graphs or chordal graphs. Linear time algorithms for computing $\gamma_t^-(T)$ of an arbitrary tree $T$ are also presented in the work. Let $f: V \to \{-1, 0, 1\}$ be a function which assigns to each vertex of graph $G = (V, E)$ an element of the set $\{-1, 0, 1\}$. We define the function $f$ to be *minus total dominating function* (MTDF) of $G$ if $f[v] \geqslant 1$ for every $v \in V$. An MTDF $f$ is *minimal* if every MTDF $g$ satisfying $g(v) \leqslant f(v)$ for every $v \in V$, is equal to $f$. The *minus total domination number*, denoted by $\gamma_t^-(G)$, of $G$ is the minimum weight of an MTDF on $G$. The *upper minus total domination number*, denoted by $\Gamma_t^-(G)$, of $G$ is the maximum weight of a minimal MTDF on $G$. We call a minimal MTDF of weight $\Gamma_t^-(G)$ a $\Gamma_t^-$-function on $G$. For a vertex $v \in V$, if $f[v] = 1$, then $v$ is called a *critical vertex* under $f$.

Throughout this paper, if $f$ is a $\Gamma_t^-(G)$-function on $G$, then we let $P$, $Q$ and $M$ denote the sets of those vertices in $G$ which are assigned under $f$ the value $+1$, 0 and $-1$, respectively. Furthermore, we define

$$P_{ij} = \{v \in P \mid d_Q(v) = i, d_M(v) = j\},$$

$$Q_{ij} = \{v \in Q \mid d_P(v) = i, d_M(v) = j\},$$

$$M_{ij} = \{v \in M \mid d_P(v) = i, d_Q(v) = j\},$$

and let $|P| = p$, $|Q| = q$ and $|M| = m$. Thus, $n = p + q + m$, $w(f) = |P| - |M| = n - q - 2m$.

The motivation described in [5] for studying this variation of the minus total domination number is rich and varied from a modeling. The following example provides an illustrative background of such an application. Consider a network of people or organizations in which some global decisions must be made in terms of preferences, such as negative, neutral or positive response. We can assign value $+1$ to vertices (individuals) of positive opinion, 0 to vertices of no opinion and $-1$ to vertices of negative opinion, of the graph. We further assume that an individual's vote is affected by the opinions of neighboring individuals, and the individual gives equal weight to the opinions of neighboring individuals. This assumption allows those individuals of high degree have greater "influence". A voter votes "YES" if there are more vertices in its neighborhood with positive opinion than those with negative opinion, and votes "NO" otherwise. For such a model, we look for an assignment of opinions that guarantee an unanimous decision; that is, for which every vertex votes YES. We call such an assignment of opinions, if available, an uniformly positive assignment. Among all uniformly positive assignments of opinions, we are primarily interested in the minimum number of vertices (individuals) who have a positive or neutral opinion. The minus total domination number is the minimum possible sum of all opinions, with $-1$ for a negative opinion, 0 for a neutral opinion and $+1$ for a positive opinion, in a uniformly positive assignment of opinions. Therefore, the minus total domination number represents the minimum number of individuals which can have positive or neutral opinions and in doing so force every individual to vote YES.

In this paper, we establish sharp upper bounds on $\Gamma_t^-(G)$ for a cubic graph and a 4-regular graph in terms of their order and characterize the graphs attaining these upper bounds.

## 2. The cubic graph

In this section we establish an upper bound on the upper minus total domination number of a cubic graph in terms of its order and characterize the cubic graphs attaining this bound.

For this purpose, we define a family $\mathscr{T}=\{G_{k,l} \mid k\geqslant 1, l\geqslant 0\}$ of cubic graphs as follows. For two integers $k\geqslant 1, l\geqslant 0$, let $G_{k,l}$ be a cubic graph with vertex set $\bigcup_{i=1}^{5} A_i$ with $|A_i|=a_i$, $i=1,2,\ldots,5$ where all $a_i$'s are integers satisfying $a_1=2k,\ a_2=2l,\ a_3=3a_1=6k,\ a_4=2a_2=4l$ and $a_5=a_3+2a_4=6k+8l$, and $A_1$ and $A_4$ are two independent sets. The edge set of $G_{k,l}$ is constructed as follows.

Add $3a_1$ edges between $A_1$ and $A_3$ so that each vertex in $A_1$ has degree 3 while each vertex in $A_3$ has degree 1. Add $3k$ edges joining vertices of $A_3$ so that $A_3$ induces a 1-regular graph. Add $l$ edges joining vertices of $A_2$ so that $A_2$ also induces a 1-regular graph. Add $2a_2$ edges between $A_2$ and $A_4$ so that each vertex in $A_2$ has degree 3 while each vertex in $A_4$ has degree 1. Add $a_5(=a_3+2a_4)$ edges between $A_3\cup A_4$ and $A_5$ in such a way that each vertex of $A_5$ is adjacent to precisely a vertex of $A_3\cup A_4$, and each vertex in $A_3$ is adjacent to precisely one vertex of $A_5$ while each vertex of $A_4$ is adjacent to precisely two vertices of $A_5$. Finally, add $a_5$ edges joining vertices of $A_5$ so that $A_5$ induces a 2-regular graph. By our construction, $G_{k,l}$ is a cubic graph of order $n=14(k+l)$. Fig. 1 shows the graph $G_{1,1}$.

By definition, the following observation is straightforward.

**Observation 1.** *A* MTDF *on a graph $G=(V,E)$ is minimal if and only if for every vertex $v\in V$ with $f(v)\geqslant 0$, there exists a vertex $u\in N(v)$ with $f[u]=1$.*

**Theorem 2.** *If $G$ is a cubic graph of order $n$, then*

$$\Gamma_t^-(G)\leqslant \tfrac{5}{7}n.$$

*The equality holds if and only if $G\in\mathscr{T}$.*

**Proof.** Let $f$ be a $\Gamma_t^-$-function on $G$. Then $\Gamma_t^-(G)=|P|-|M|=p-m$. By definition, for any vertex $v\in V$, $d_M(v)\leqslant 1$, $d_Q(v)\leqslant 2-2d_M(v)$ and $d_P(v)\geqslant d_M(v)+1\geqslant 1$ for otherwise $f[v]<1$. Hence we can partition $P$, $Q$ and $M$ into the following sets, respectively.

$$P_{ij}=\{v\in P \mid d_Q(v)=i, d_M(v)=j,\ \text{where } 0\leqslant j\leqslant 1, 0\leqslant i\leqslant 2-2j\},$$

$$Q_{ij}=\{v\in Q \mid d_P(v)=i, d_M(v)=j,\ \text{where } 0\leqslant j\leqslant 1,\ j+1\leqslant i\leqslant 3-j\},$$

$$M_{ij}=\left\{v\in M \mid d_P(v)=i, d_Q(v)=j,\ \text{where } 0\leqslant j\leqslant 2, \left\lfloor\frac{3-j}{2}\right\rfloor+1\leqslant i\leqslant 3-j\right\},$$

and let $|P_{ij}|=p_{ij}$, $|Q_{ij}|=q_{ij}$, and $|M_{ij}|=m_{ij}$. Then $p=p_{00}+p_{01}+p_{10}+p_{20}$, $q=q_{10}+q_{20}+q_{21}+q_{30}$, $m=m_{12}+m_{20}+m_{21}+m_{30}$. Furthermore, we write

$$P'=P_{01}\cup P_{20},\quad Q'=Q_{10}\cup Q_{21},\quad M'=M_{12}\cup M_{20}.$$

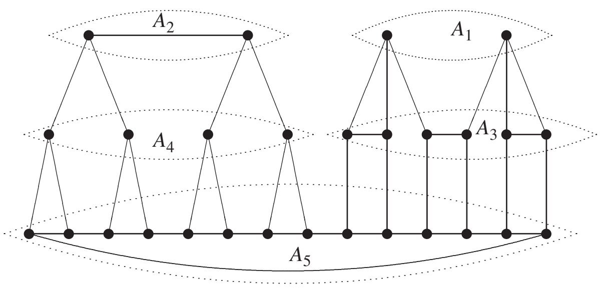


Fig. 1. The graph $G_{1,1}$.

Clearly, each vertex $v$ in $P' \cup Q' \cup M'$ is a critical vertex of $G$ under $f$, i.e. $f[v] = 1$, while for each vertex $v \in V - (P' \cup Q' \cup M')$, $f[v] \geqslant 2$. By counting the edge number $e(Q, M)$, $e(P, M)$ and $e(P, Q)$, we immediately get the following equalities:

$$q_{21} = e(Q, M) = 2m_{12} + m_{21}, \tag{1}$$

$$p_{01} = e(P, M) = 3m - (q_{21} + m_{20}), \tag{2}$$

and

$$p_{10} + 2p_{20} = e(P, Q) = 3q - (2q_{10} + q_{20} + q_{21}), \tag{3}$$

or equivalently

$$p_{10} + 2p_{20} = q_{10} + 2q_{20} + 2q_{21} + 3q_{30}. \tag{4}$$

By Observation 1, for every vertex $v \in P - P' = P_{00} \cup P_{10}$, there exists a vertex $u \in N(v)$ such that $f[u] = 1$. It follows that for every vertex $v \in P_{00}$, there must exist a neighbor of $v$ that belongs to $P'$, while for every vertex $v \in P_{10}$, there must exist a neighbor of $v$ that belongs to $P' \cup Q'$. Therefore,

$$\begin{aligned} p_{00} + p_{10} &\leqslant e(P_{00} \cup P_{10}, P' \cup Q') \\ &\leqslant e(P_{00} \cup P_{10}, P_{01}) + e(P_{00} \cup P_{10}, P_{20} \cup Q'). \end{aligned}$$

Furthermore, we note that for every vertex $v \in P_{01}$, there must exist a neighbor $v'$ of $v$ satisfying $f[v'] = 1$, that is, $v' \in P' \cup M'$. If $v' \in P'$, then $v$ is adjacent to at most a vertex of $P_{00} \cup P_{10}$, while if $v' \in M'$, then $v$ is adjacent to at most two vertices of $P_{00} \cup P_{10}$. Hence, we can write $P_{01}$ as the disjoint union of two sets $P'_{01}$ and $P''_{01}$ where $P'_{01} = \{v \in P_{01} \mid d_{P_{00} \cup P_{10}}(v) = 2\}$ and $P''_{01} = P_{01} - P'_{01}$. Let $|P'_{01}| = p'_{01}$, and so $|P''_{01}| = p_{01} - p'_{01}$. Since each vertex $v \in P'_{01}$ is adjacent to precisely one vertex in $M'$, it follows that

$$p'_{01} \leqslant e(P'_{01}, M') = e(P'_{01}, M_{12} \cup M_{20}) \leqslant m_{12} + 2m_{20}.$$

So

$$\begin{aligned} e(P_{00} \cup P_{10}, P_{01}) &= e(P_{00} \cup P_{10}, P'_{01} \cup P''_{01}) \\ &\leqslant 2p'_{01} + (p_{01} - p'_{01}) \\ &= p_{01} + p'_{01} \\ &\leqslant p_{01} + m_{12} + 2m_{20}. \end{aligned}$$

Similarly, it follows that for every vertex $v \in P_{20}$, there must exist a neighbor $v'$ of $v$ that belongs to $P' \cup Q'$. If $v' \in P'$, then $v$ has no neighbor in $P_{00} \cup P_{10}$, while if $v' \in Q'$, then $v$ is adjacent to at most a vertex of $P_{00} \cup P_{10}$. We partition $P_{20}$ into two subsets $P'_{20} = \{v \in P_{20} \mid d_{P_{00} \cup P_{10}}(v) = 1\}$ and $P''_{20} = P_{20} - P'_{20}$. Let $|P'_{20}| = p'_{20}$, and so $|P''_{20}| = p_{20} - p'_{20}$. Since each vertex $v \in P'_{20}$ is adjacent to a vertex in $Q'$, it follows that $p'_{20} \leqslant e(P'_{20}, Q')$. So we have

$$\begin{aligned} e(P_{00} \cup P_{10}, P_{20} \cup Q') &= e(P_{00} \cup P_{10}, P'_{20} \cup P''_{20}) + e(P_{00} \cup P_{10}, Q') \\ &\leqslant p'_{20} + e(P_{00} \cup P_{10}, Q') \\ &\leqslant e(P'_{20}, Q') + e(P_{00} \cup P_{10}, Q') \\ &= e(P_{00} \cup P_{10} \cup P'_{20}, Q_{10} \cup Q_{21}) \\ &\leqslant q_{10} + 2q_{21}. \end{aligned}$$

Thus,

$$p_{00} + p_{10} \leqslant p_{01} + (m_{12} + 2m_{20}) + (q_{10} + 2q_{21}). \tag{5}$$

Using the above equalities and inequalities, we next complete the proof of the theorem.

$$\begin{aligned} n &= (q+m)+p \\ &= (q+m)+(p_{00}+p_{10})+(p_{01}+p_{20}) \\ &\leqslant (q+m)+2p_{01}+(p_{10}+2p_{20})-p_{10}-p_{20}+(m_{12}+2m_{20})+(q_{10}+2q_{21}) \quad \text{(by (5))} \\ &\leqslant 7(q+m)-a, \end{aligned}$$

where $a = 2p_{10}+3p_{20}+3q_{10}+2q_{20}+2q_{21}-m_{12}$. The last inequality comes from (2) and (3). We obtain

$$q+m \geqslant \tfrac{1}{7}n+\tfrac{1}{7}a.$$

So

$$p = n-(q+m) \leqslant \tfrac{6}{7}n-\tfrac{1}{7}a. \tag{6}$$

On the other hand, combining (2) and (6), we immediately get

$$\begin{aligned} p &= (p_{00}+p_{10})+(p_{01}+p_{20}) \\ &\leqslant 2p_{01}+p_{20}+(m_{12}+2m_{20})+(q_{10}+2q_{21}) \\ &\leqslant 6m+(p_{20}+q_{10}+m_{12}), \end{aligned}$$

which implies

$$m \geqslant \tfrac{1}{6}p-\tfrac{1}{6}(p_{20}+q_{10}+m_{12}). \tag{7}$$

Therefore,

$$\begin{aligned} \Gamma_t^-(G) &= p-m \\ &\leqslant \tfrac{5}{6}p+\tfrac{1}{6}(p_{20}+q_{10}+m_{12}) \quad \text{(by (7))} \\ &\leqslant \tfrac{5}{7}n-\tfrac{1}{21}[(5p_{10}+4p_{20}+4q_{10}+5q_{20}+4q_{21})-6m_{12}] \quad \text{(by (6))} \\ &\leqslant \tfrac{5}{7}n-\tfrac{1}{21}(5p_{10}+4p_{20}+4q_{10}+5q_{20}+2q_{21}+3m_{21}) \quad \text{(by (1))} \\ &\leqslant \tfrac{5}{7}n. \end{aligned}$$

For a cubic graph $G$ of order $n$, we next show that if $\Gamma_t^-(G)=5n/7$, then $G \in \mathscr{T}$. Suppose that $\Gamma_t^-(G)=5n/7$, then equalities hold for the above inequalities, so $p_{10}=p_{20}=0$, $q_{10}=q_{20}=q_{21}=0$, $m_{21}=0$. Furthermore, equality (3) implies that $Q=\emptyset$, and by equality (1), $m_{12}=0$. Hence, we have $V=P_{00}\cup P_{01}\cup M_{20}\cup M_{30}$, and the equalities from (6) and (7), it follows that $m=p/6$ and $p=6n/7$. By (2) and the equality from (5), we get

$$\begin{aligned} p_{01} &= 2m_{20}+3m_{30}, \\ p_{00} &= p_{01}+2m_{20} = 4m_{20}+3m_{30}. \end{aligned}$$

Obviously, $P_{01}$ is partitioned into subsets $N(M_{20})$ and $N(M_{30})$. Note that each component of $G[M_{20}]$ is isomorphic to $K_2$. Hence, $m_{20}$ is even. Let $m_{20}=2l$ ($\geqslant 0$). Moreover, for each $v \in N(M_{30})$, $v$ is adjacent to at most a vertex in $P_{00}$, for otherwise there is no neighbor $u$ of $v$ such that $f[u]=1$, which contradicting Observation 1. Hence

$$\begin{aligned} e(P_{00},P_{01}) &= e(P_{00}, N(M_{20})\cup N(M_{30})) \\ &\leqslant 2|N(M_{20})|+|N(M_{30})| \\ &\leqslant 4m_{20}+3m_{30}. \end{aligned}$$

On the other hand, by minimality of $f$, each vertex $v \in P_{00}$ has at least a neighbor that belongs to $P_{01}$. Hence $e(P_{00}, P_{01}) \geqslant p_{00} = 4m_{20} + 3m_{30}$. Therefore, we have

$$e(P_{00}, P_{01}) = 4m_{20} + 3m_{30}.$$

This means that each vertex of $N(M_{20})$ is adjacent to precisely two vertices of $P_{00}$, and each vertex of $N(M_{30})$ is adjacent to precisely a vertex of $P_{00}$. Hence, each vertex of $P_{00}$ has precisely a neighbor that belongs to $P_{01}$ and each component of $G[P_{00}]$ is 2-regular. Let $N(M_{20}) = P'_{01}$, $N(M_{30}) = P''_{01}$. Then $P'_{01}$ is an independent set in $G[P_{01}]$, while each component of $G[P''_{01}]$ is isomorphic to $K_2$. Hence $|N(M_{30})| = 3m_{30}$ is even, which implies that $m_{30}$ is even. Let $m_{30} = 2k$. Thus, $G = G_{k,l}$ with vertex set $\bigcup_{i=1}^{5} A_i$, where $A_1 = M_{30}$, $A_2 = M_{20}$, $A_3 = N(M_{30}) = P''_{01}$, $A_4 = N(M_{20}) = P'_{01}$ and $A_5 = P_{00}$. Consequently, $G \in \mathscr{T}$.

Conversely, suppose that $G \in \mathscr{T}$. Let $G = G_{k,l}$ for integers $k \geqslant 1$, $l \geqslant 0$. The function $f$ that assigns to each vertex of $A_1 \cup A_2$ the value $-1$ and to all other vertices the value $+1$ is a minimal MTDF on $G$ with weight $w(f) = p - m = [(6k + 8l) + (6k + 4l)] - 2(k + l) = 10(k + l) = 5n/7$. Consequently, $\Gamma_t^-(G) = 5n/7$. □

In general, we do not know whether the parameters $\Gamma_t^-$ and $\Gamma_t^s$ are comparable. However, since every signed TDF is also an MTDF, we have the following result.

**Theorem 3.** *If G is a graph with all odd vertices, then* $\Gamma_t^s(G) \leqslant \Gamma_t^-(G)$.

**Proof.** Let $f$ be a $\Gamma_t^s$-function on $G$. Let $P$ be the set of vertices of weight +1 and $M$ the set of vertices of weight $-1$. Since $f$ is minimal, it follows that every vertex $v \in P$ has at least a neighbor $u$ such that $f[u] = 1$ or 2. Clearly, for every vertex $v \in V$, $f[v] = d(v) - 2d_M(v)$. This implies that $f[u] = 1$. Thus, $f$ is a minimal MTDF of $G$. So $\Gamma_t^s(G) = w(f) \leqslant \Gamma_t^-(G)$. □

As an immediate consequence of Theorems 2 and 3, we get the special case of a result due to Henning [7].

**Corollary 4** (*Henning [7]*). *If G be a cubic graph of order n, then* $\Gamma_t^s(G) \leqslant \frac{5}{7}n$.

Finally, we get the following result on upper minus total domination of a cubic graph.

**Theorem 5.** *Let G be a cubic graph of order n. Then the following statement are equivalent*:

(1) $\Gamma_t^s(G) = \frac{5}{7}n$;
(2) $\Gamma_t^-(G) = \frac{5}{7}n$;
(3) $G \in \mathscr{T}$.

## 3. The 4-regular graph

In this section we turn our attention to 4-regular graphs. We present an upper bound on the upper minus total domination number of a 4-regular graph and characterize the 4-regular graphs attaining this bound.

To complete our characterization, we first construct a family $\mathscr{F}$ of 4-regular graphs. For two integers $k \geqslant 1$, $0 \leqslant l \leqslant k$, let $H_{k,l}$ be a graph with vertex set $\bigcup_{i=1}^{8} A_i$ with $|A_i| = a_i$, for $1 \leqslant i \leqslant 8$, where all $a_i$'s are integers satisfying $a_1 = a_3 = 2l$, $a_2 = a_4 = k$, $a_5 = a_6 = 4l$, $a_7 = 4(k - l)$ and $a_8 = 4(k + 3l)$, and where $A_2$, $A_4$, $A_5$ and $A_6$ are independent sets. The edge set of $H_{k,l}$ is constructed as follows.

Add $l$ edges joining vertices of $A_1$ (resp. $A_3$) so that $A_1$ (resp. $A_3$) induces a 1-regular graph. Add $2(k - l)$ edges joining vertices of $A_7$ so that $A_7$ induces a 1-regular graph also. Add $6(k + 3l)$ edges joining vertices of $A_8$ so that $A_8$ induces a 3-regular graph. Add $2l$ edges between $A_1$ and $A_3$ so that each vertex in $A_1$ is adjacent to precisely one vertex of $A_3$ and each vertex in $A_3$ is also adjacent to precisely one vertex of $A_1$, so each vertex of $A_1 \cup A_3$ has degree 2. Add $4l$ edges between $A_1$ (resp. $A_3$) and $A_5$ (resp. $A_6$) so that each vertex in $A_1$ (resp. $A_3$) is adjacent to two vertices of $A_5$ (resp. $A_6$) and each vertex of $A_5$ (resp. $A_6$) is adjacent to one vertex of $A_1$ (resp. $A_3$), so each vertex in $A_1 \cup A_3$ has degree 4 while each vertex in $A_5 \cup A_6$ has degree 1. Add $4k$ edges between $A_2$ (resp. $A_4$) and $A_6 \cup A_7$ (resp. $A_5 \cup A_7$) so that each vertex of $A_2 \cup A_4$ has degree 4 while each vertex of $A_6$ (resp. $A_5$) is adjacent to a vertex of $A_2$

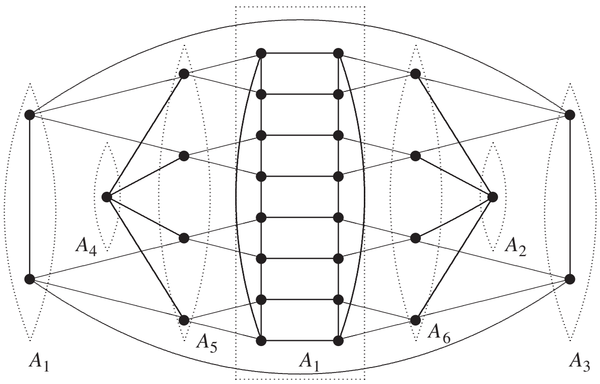


Fig. 2. The graph $H_{1,1}$.

(resp. $A_4$) and each vertex of $A_7$ is, respectively, adjacent to a vertex of $A_2$ and $A_4$. Then each vertex in $A_5 \cup A_6$ has degree 2 while each vertex in $A_7$ has degree 3. Finally, add $4(k+3l)$ edges between $A_5 \cup A_6 \cup A_7$ and $A_8$ in such a way that each vertex $A_5 \cup A_6$ is adjacent to precisely two vertices of $A_8$, and each vertex of $A_7$ is adjacent to precisely one vertex of $A_8$ while each vertex of $A_8$ is adjacent to precisely one vertex of $A_5 \cup A_6 \cup A_7$.

By the construction, each vertex in $H_{k,l}$ has degree 4. So $H_{k,l}$ is a 4-regular graph with order $n=\sum_{i=1}^{8} a_i = 10(k+2l)$. Fig. 2 shows the graph $H_{1,1}$.

**Theorem 6.** *If G is a* 4*-regular graph of order n*, *then*

$$\Gamma_t^-(G) \leqslant \tfrac{7}{10} n.$$

*The equality holds if and only if* $G \in \mathscr{F}$.

**Proof.** Let $f$ be a $\Gamma_t^-(G)$-function on $G$. Similar to Theorem 2, we can respectively, partition $P$, $Q$ and $M$ into six subsets as follows:

$$P = P_{00} \cup P_{01} \cup P_{10} \cup P_{11} \cup P_{20} \cup P_{30},$$

$$Q = Q_{10} \cup Q_{20} \cup Q_{21} \cup Q_{30} \cup Q_{31} \cup Q_{40},$$

$$M = M_{13} \cup M_{21} \cup M_{22} \cup M_{30} \cup M_{31} \cup M_{40}.$$

Let $P' = P_{11} \cup P_{30}$, $Q' = Q_{10} \cup Q_{21}$ and $M' = M_{13} \cup M_{21}$. Clearly, $P' \cup Q' \cup M'$ is the set of all critical vertices of $G$ under $f$, that is, for each vertex $v \in P' \cup Q' \cup M'$, $f[v] = 1$, while each vertex $v$ in $V - (P' \cup Q' \cup M')$, $f[v] \geqslant 2$. Again by counting the edge number $e(Q, M)$, $e(P, M)$ and $e(P, Q)$, we get the following equalities:

$$\begin{aligned} q_{21} + q_{31} &= e(Q, M) \\ &= m_{31} + 2m_{22} + 3m_{13} + m_{21}, \end{aligned} \tag{8}$$

$$\begin{aligned} p_{01} + p_{11} &= e(P, M) \\ &= 4m - (m_{31} + 2m_{22} + 3m_{13} + 2m_{21} + m_{30}) \\ &= 4m - (q_{31} + q_{21} + m_{21} + m_{30}), \end{aligned} \tag{9}$$

and

$$\begin{aligned} p_{10} + p_{11} + 2p_{20} + 3p_{30} &= e(P, Q) \\ &= 4q - (3q_{10} + 2q_{20} + 2q_{21} + q_{30} + q_{31}), \end{aligned} \tag{10}$$

or equivalently

$$4q = p_{10} + p_{11} + 2p_{20} + 3p_{30} + 3q_{10} + 2q_{20} + 2q_{21} + q_{30} + q_{31}. \tag{11}$$

By Observation 1, for each vertex $v \in P - P'$, there must exist a neighbor $u$ of $v$ such that $f[u] = 1$. That is, $u \in P' \cup Q' \cup M'$. Hence, we have

$$\begin{aligned} p_{00} + p_{01} + p_{10} + p_{20} &\leqslant e(P - P', P') \\ &= e(P - P', P_{11}) + e(P - P', P_{30}) \\ &\quad + e(P - P', Q_{10} \cup Q_{21}) + e(P - P', M_{13} \cup M_{21}). \end{aligned}$$

Furthermore, it follows that for every vertex $v \in P_{11}$, there must exist a critical neighbor $v'$ of $v$, i.e., $v' \in P' \cup Q' \cup M'$. If $v' \in P'$, then $v$ is adjacent to at most one vertex of $P - P'$, while if $v \in Q' \cup M'$, then $v$ is adjacent to at most two vertices of $P - P'$. Hence, we can write $P_{11}$ as the disjoint union of two sets $P'_{11} = \{v \in P_{11} | \ d_{P-P'}(v) = 2\}$, $P''_{11} = P_{11} - P'_{11}$. Let $P'_{11} = p'_{11}$, and so $|P''_{11}| = p_{11} - p'_{11}$. Note that every vertex $v \in P'_{11}$ is, respectively, adjacent to a vertex of $Q'$ and $M'$. Then

$$\begin{aligned} p'_{11} &\leqslant e(P'_{11}, Q_{10} \cup Q_{21} \cup M_{13} \cup M_{21}) \\ &= e(P'_{11}, Q_{10} \cup Q_{21}) + e(P'_{11}, M_{13} \cup M_{21}). \end{aligned}$$

So

$$\begin{aligned} e(P - P', P_{11}) &= e(P - P', P'_{11} \cup P''_{11}) \\ &\leqslant 2p'_{11} + (p_{11} - p'_{11}) \\ &\leqslant p_{11} + e(P'_{11}, Q_{10} \cup Q_{21}) + e(P'_{11}, M_{13} \cup M_{21}). \end{aligned} \tag{12}$$

Similarly, by the minimality of $f$, for each vertex $v \in P_{30}$, there must exist a critical neighbor $v'$ of $v$. If $v' \in P'$, then $v$ is adjacent to no vertex of $P - P'$, while if $v \in Q' \cup M'$, then $v$ is adjacent to at most one vertex of $P - P'$. Hence, we can write $P_{30}$ as the disjoint union of two sets $P'_{30} = \{v \in P_{30} | \ d_{P-P'}(v) = 1\}$, $P''_{30} = P_{30} - P'_{30}$. Let $P'_{30} = p'_{30}$, and so $|P''_{30}| = p_{30} - p'_{30}$. Since every vertex $v \in P'_{30}$ is adjacent to a vertex of $Q'$, it follows that

$$\begin{aligned} e(P - P', P_{30}) &= e(P - P', P'_{30} \cup P''_{30}) \\ &= e(P - P', P'_{30}) \\ &= p'_{30} \\ &\leqslant e(P'_{30}, Q_{10} \cup Q_{21}). \end{aligned} \tag{13}$$

Thus, by (12) and (13), we get

$$\begin{aligned} p_{00} + p_{01} + p_{10} + p_{20} &\leqslant p_{11} + e(P'_{11}, Q_{10} \cup Q_{21}) + e(P'_{11}, M_{13} \cup M_{21}) \\ &\quad + e(P'_{30}, Q_{10} \cup Q_{21}) + e(P - P', Q_{10} \cup Q_{21}) + e(P - P', M_{13} \cup M_{21}) \\ &\leqslant p_{11} + (q_{10} + 2q_{21}) + (m_{13} + 2m_{21}). \end{aligned} \tag{14}$$

Next, we start to establish the upper bound on $\Gamma_t^-(G)$. First, we obtain

$$\begin{aligned} n &= (q + m) + p \\ &= (q + m) + (p_{00} + p_{01} + p_{10} + p_{20}) + (p_{11} + p_{30}) \\ &\leqslant (q + m) + p_{11} + (p_{11} + p_{30}) + (q_{10} + 2q_{21}) + (m_{13} + 2m_{21}) \quad \text{(by (14))} \\ &\leqslant 5(q + m) - b_1 \quad \text{(by (9), (10))}, \end{aligned}$$

where $b_1 = p_{01} + p_{10} + 2p_{20} + 2p_{30} + 2q_{10} + 2q_{20} + q_{21} + q_{30} + 2q_{31} + m_{30} - m_{13} - m_{21}$. This implies that

$$q + m \geqslant \tfrac{1}{5}n + \tfrac{1}{5}b_1.$$

So

$$p = n - (q + m) \leqslant \tfrac{4}{5}n - \tfrac{1}{5}b_1. \tag{15}$$

On the other hand, we have

$$\begin{aligned} p &= (p_{00} + p_{01} + p_{10} + p_{20}) + (p_{11} + p_{30}) \\ &\leqslant 2p_{11} + p_{30} + (q_{10} + 2q_{21}) + (m_{13} + 2m_{21}) \quad \text{(by (14))} \\ &\leqslant 8m - 2(p_{01} + q_{31} + m_{30}) + (p_{30} + q_{10} + m_{13}). \end{aligned}$$

The last inequality comes from (9). Then, we obtain

$$m \geqslant \tfrac{1}{8}p + \tfrac{1}{8}[2(p_{01} + q_{31} + m_{30}) - (p_{30} + q_{10} + m_{13})]. \tag{16}$$

Consequently, by (15) and (16),

$$\begin{aligned} \Gamma_t^-(G) &= p - m \\ &\leqslant \tfrac{7}{8}p - \tfrac{1}{8}[2(p_{01} + q_{31} + m_{30}) - (p_{30} + q_{10} + m_{13})] \\ &\leqslant \tfrac{7}{10}n - \tfrac{1}{40}(17p_{01} + 7p_{10} + 14p_{20} + 9p_{30} + 9q_{10} + 14q_{20} \\ &\quad + 7q_{30} + 7q_{21} + 24q_{31} + 17m_{30} - 12m_{13} - 7m_{21}) \\ &\leqslant \tfrac{7}{10}n - \tfrac{1}{40}b_2 \quad \text{(by (8))} \\ &\leqslant \tfrac{7}{10}n, \end{aligned}$$

where $b_2 = (17p_{01} + 7p_{10} + 14p_{20} + 9p_{30} + 9q_{10} + 14q_{20} + 7q_{30} + 17q_{31} + 9m_{13} + 17m_{30} + 14m_{22} + 7m_{31})$.

If $\Gamma_t^-(G) = 7n/10$, then equalities hold for the above inequalities. By $b_2 = 0$, we immediately have

$$\begin{aligned} &p_{01} = p_{10} = p_{20} = p_{30} = 0, \\ &q_{10} = q_{20} = q_{30} = q_{31} = 0, \\ &m_{13} = m_{22} = m_{30} = m_{31} = 0. \end{aligned}$$

And by the equalities from (15) and (16), it follows that $p = \frac{4}{5}n$, $q = m = \frac{1}{10}n$. Applying equalities (8), (9), (11) and the equality from (14), we obtain

$$\begin{aligned} &q_{21} = m_{21}, \\ &p_{11} = 4q_{40} + 2q_{21} = 4m_{40} + 2m_{21}, \\ &p_{00} = p_{11} + 2(q_{21} + m_{21}). \end{aligned}$$

Since each component of the induced subgraphs $G[Q_{21}]$ and $G[M_{21}]$ is isomorphic to $K_2$, it follows that $q_{21}$ and $m_{21}$ are even. Write $q_{21} = m_{21} = 2l$, $q_{40} = m_{40} = k$. Thus,

$$p_{11} = 4(k + l), \quad p_{00} = p_{11} + 4q_{21} = 4(k + 3l).$$

Furthermore, by the equality from (12), we have $p_{00} = e(P - P', P_{11}) = p_{11} + p'_{11}$. Hence, $p'_{11} = p_{00} - p_{11} = 4q_{21} = 8l$, and so $p''_{11} = p_{11} - p'_{11} = 4(k - l)$. Since every vertex in $P''_{11}$ has precisely a neighbor that belongs to $P_{00}$, it implies that $G[P''_{11}]$ is 1-regular subgraph of $G$. Moreover, note that $P'_{11}$ is an independent set of vertices of $G[P]$. By Observation 1, every vertex $v$ of $P'_{11}$ has at least a neighbor $u$ of $v$ such that $f[u] = 1$. Thus, if $v$ has a neighbor that belongs to $Q_{40}$, then

$v$ must have a neighbor that belongs to the critical vertex set $M_{21}$. Similarly, if $v$ has a neighbor that belongs to $M_{40}$, then $v$ must have a neighbor that belongs to the critical vertex set $Q_{21}$. So we have that $e(P'_{11}, Q_{40}) = e(P'_{11}, M_{21}) = 4l$ and $e(P'_{11}, M_{40}) = e(P'_{11}, Q_{21}) = 4l$. Write $M_{21} = A_1$, $M_{40} = A_2$, $Q_{21} = A_3$, $Q_{40} = A_4$, $P_{00} = A_8$, $P''_{11} = A_7$ and $P'_{11} = A_5 \cup A_6$, where each vertex of $A_5$ has a neighbor that belongs to $Q_{40}$ while $A_6$ has a neighbor that belongs to $M_{40}$. Therefore, $G \in \mathscr{F}$.

Conversely, suppose that $G \in \mathscr{F}$. Thus, there exist two integers $k \geqslant 1$, $0 \leqslant l \leqslant k$ such that $G = H_{k,l}$ is a 4-regular graph of order $10(k + 2l)$. Let $f$ be a function on $H_{k,l}$ which assigns to every vertex of $A_1 \cup A_2$ and $A_3 \cup A_4$ the value $-1$ and 0, respectively, and to all vertices of $\bigcup_{i=5}^{8} A_i$ the value $+1$. Then the set $A_1 \cup A_3 \cup A_5 \cup A_6 \cup A_7$ is the critical vertex set of $H_{k,l}$ under $f$. This implies that for every vertex $v \in V$, there exists a vertex $u \in N(v)$ such that $f[u] = 1$. So, $f$ is a minimal MTDF with weight $w(f) = \sum_{i=5}^{8} a_i - (a_1 + a_2) = 8(k + 2l) - (k + 2l) = 7(k + l) = 7n/10$. Consequently, $\Gamma_t^-(G) = 7n/10$. $\square$

## 4. Conclusion and open problems

The minus total domination problem in graphs is a variant of the traditional domination problem, where each vertex $v$ is assigned value $-1$ or 0 or $+1$ such that the sum of labels in each $N(v)$ is positive. From the point view of the purely graph theory, it is clear that the minus total domination problem can be seen as a proper generalization of the classical total domination problem and minus domination problem. In this paper we study upper minus total domination in small-degree regular graphs. However, the cases for 1-regular and 2-regular graphs are omitted, since their value of $\Gamma_t^-$ can be easily determined. The work is inspired by some results of NP-complete for minus total domination in [5]. We establish the upper bounds on $\Gamma_t^-$ for 3-regular graphs and 4-regular graphs and give complete characterization of graphs achieving these upper bounds.

However, the study also make us to believe that finding a sharp upper bound of $\Gamma_t^-$ in general graphs is rather difficult. The paper is thus closed by stating several open problems:

1. Find the upper bounds on $\Gamma_t^-(G)$ for a $k$-regular graph $G$, $k \geqslant 5$.
2. For any positive integer $k$, does there exist a family of graphs satisfying $\Gamma_t^- - \gamma_t^s \geqslant k$?
3. Is it true that if $G$ is a 4-regular graph, then $\Gamma_t^-(G) \leqslant \Gamma_t^s(G)$?

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